

## Chapter 4. Electromagnetism and Maxwell's Equations

### Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 6.*

### 4.1 Maxwell's Displacement Current

Of the four equations derived so far

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\end{aligned}\tag{4.1}$$

the third one (i.e., Ampère's law) is not consistent with what we know to be true of time-dependent fields. This fact should not be too surprising since Ampère's law was derived specifically for steady-state current configurations. More precisely, the condition imposed on the divergence of the current density (i.e.,  $\nabla \cdot \mathbf{J} = 0$ ) can be explicitly verified from

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{H}) = 0,\tag{4.2}$$

since the divergence of the curl is always zero. However, we know (see equation (3.2)) that the divergence of the current density will not vanish for systems where the charge density is time-dependent; the so-called **continuity equation** for charge and current is then applicable

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.\tag{4.3}$$

However, since the charge density  $\rho$  can be evaluated from the electric displacement vector  $\mathbf{D}$  from the first of equations (4.1), we can transform equation (4.3) to

$$\nabla \cdot \left[ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] = 0.\tag{4.4}$$

It was Maxwell's idea that by substituting  $\mathbf{J}$  with  $\mathbf{J} + \partial \mathbf{D} / \partial t$  in Ampère's law (and therefore modifying it) the aforementioned inconsistency could be erased. He then proposed that the four equations (4.1) be replaced by a new set of equations that would apply to time varying fields. That is, the modified set

$$\begin{array}{l}
\nabla \cdot \mathbf{D} = \rho \\
\nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\end{array}
\tag{4.5}$$

is famously called **Maxwell's equations**. The new term added by Maxwell is called the **displacement current** and is responsible, with the  $\partial \mathbf{B} / \partial t$  term present in Faraday's law, for the propagation of electromagnetic waves. It is also important to note, however, that for static fields, i.e., more precisely when  $\partial \mathbf{D} / \partial t = 0$ , all the steady-state experimental phenomena investigated by Biot and Savart, and Ampère are still accounted for with equations (4.5). Equally important for the solution of problems in electromagnetism is the fact that the boundary conditions derived for the four fields in chapters 1, 2 and 3 for steady-state situations are still valid in for time varying fields. That is,

$$\begin{array}{l}
\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \\
\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \\
\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \\
\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}
\end{array}
\tag{4.6}$$

for the boundary between two regions of index 1 and 2, respectively, with  $\mathbf{n}$  the unit vector normal to the boundary and directed from region 1 to region 2.

### Example

Thin wires are connected to the centres of corresponding circular plates (of radius  $a$ ), which form a capacitor. The current  $I$  carried by the wires is constant, and the separation  $w$  between the plates is such that  $w \ll a$ . Assume that the current flows out over the plates in such a way that the surface charge is uniform at any given time, and is zero at  $t = 0$ .

- a) Find the electric field between the plates, as a function of time.
- b) Find the displacement current through a circle of radius  $s < a$  in the plane midway between the plates. Using the flat surface contained within this circle, find the magnetic induction field at a distance  $s$  from the axis joining the wires.

### Solution

- a) The electric field between the plates is (see Problem 3 of the first problem list)

$$\mathbf{E}(t) = \frac{\sigma(t)}{\epsilon_0} \mathbf{e}_z, \quad (4.7)$$

where  $\sigma(t)$  is the surface charge density on a plate and  $\mathbf{e}_z$  is the unit vector parallel to the wires. If  $Q(t)$  is the charge on a plate at time  $t$ , then

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}, \quad (4.8)$$

and

$$\mathbf{E}(t) = \frac{It}{\pi \epsilon_0 a^2} \mathbf{e}_z. \quad (4.9)$$

**b)** We define the displacement current density  $\mathbf{J}_d$  with

$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (4.10)$$

and use it in the integral equation defining Ampère's Law (i.e., equation (3.23)) such that

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int (\mathbf{J} + \mathbf{J}_d) \cdot \mathbf{n} da. \quad (4.11)$$

But in this problem the electric current density  $\mathbf{J}$  between the plates is zero, and from equations (4.9) to (4.11) we have

$$\begin{aligned} 2\pi s B_\phi &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left( \frac{It}{\pi \epsilon_0 a^2} \right) \pi s^2 \\ &= \mu_0 I \frac{s^2}{a^2}, \end{aligned} \quad (4.12)$$

and

$$\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2} \mathbf{e}_\phi, \quad s < a. \quad (4.13)$$

We therefore see that Maxwell's displacement current plays the same role as that of an electric current, as it is the source of a magnetic induction field. That is, *a time varying electric field will generate a magnetic field.*

## 4.2 The Vector and Scalar Potentials and Gauge Transformations

The same vector potential  $\mathbf{A}$  introduced while dealing with magnetostatics is still valid here, since the relation  $\nabla \cdot \mathbf{B} = 0$  still holds. Therefore, we can define the magnetic induction with

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (4.14)$$

On the other hand, the relationship between the electric vector  $\mathbf{E}$  to a scalar potential  $\Phi$  cannot be as derived for electrostatics. Indeed, we must now account for Faraday's law of induction (i.e., the last of equations (4.5)), which, when combined with equation (4.14), states that

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0, \quad (4.15)$$

or, alternatively,

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}. \quad (4.16)$$

We see from this equation that the electric field cannot be simply expressed as the gradient of a scalar function, but the interdependency of the electric and magnetic induction fields must also be accounted for through the time derivative of the vector potential.

If we restrict ourselves to vacuum (i.e., we set  $\mathbf{H} = \mathbf{B}/\mu_0$  and  $\mathbf{D} = \epsilon_0 \mathbf{E}$  in Maxwell's equations), then the third equation of the set (i.e., equations (4.5)) can be transformed as follows (upon insertion of equations (4.14) and (4.16))

$$\begin{aligned} \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) + \epsilon_0 \left[ \nabla \left( \frac{\partial \Phi}{\partial t} \right) + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right] \\ &= -\frac{1}{\mu_0} \nabla^2 \mathbf{A} + \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \left( \frac{1}{\mu_0} \nabla \cdot \mathbf{A} + \epsilon_0 \frac{\partial \Phi}{\partial t} \right) \\ &= \mathbf{J}, \end{aligned} \quad (4.17)$$

or with  $c^2 = (\mu_0 \epsilon_0)^{-1}$ ,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J}. \quad (4.18)$$

In a similar fashion, Gauss' law can be written using the potentials as

$$\begin{aligned}
\varepsilon_0 \nabla \cdot \mathbf{E} &= -\varepsilon_0 \nabla \cdot \left( \nabla \Phi + \frac{\partial \mathbf{A}}{\partial t} \right) \\
&= -\varepsilon_0 \nabla^2 \Phi - \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) \\
&= \rho,
\end{aligned} \tag{4.19}$$

or

$$\nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}. \tag{4.20}$$

Just as was the case for magnetostatics, the gradient of any well-behaved scalar function  $\psi$  can be added to the potential vector without changing the magnetic induction. That is, we can choose a new potential vector  $\mathbf{A}'$  to replace the original one  $\mathbf{A}$  if

$$\mathbf{A}' = \mathbf{A} + \nabla \psi. \tag{4.21}$$

However, since the electric field must also be unchanged by this transformation, we see from equation (4.16) that the scalar potential  $\Phi$  must simultaneously be changed to a new potential  $\Phi'$  defined by

$$\Phi' = \Phi - \frac{\partial \psi}{\partial t}. \tag{4.22}$$

The availability of the additional scalar function  $\psi$  gives us a significant amount of freedom in selecting a pair of potentials  $\mathbf{A}$  and  $\Phi$  that could simplify equations (4.18) and (4.20). For example, let us suppose that given a pair  $\mathbf{A}'$  and  $\Phi'$ , we want to find new potentials  $\mathbf{A}$  and  $\Phi$  that will satisfy the so-called **Lorenz gauge**

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0. \tag{4.23}$$

Inserting equations (4.21) and (4.22) in equation (4.23) we find that

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} - \nabla^2 \psi + \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \tag{4.24}$$

Therefore, selecting the function  $\psi$  such that it satisfies equation (4.24) will ensure that the Lorenz gauge is respected, and allow for the decoupling of equations (4.18) and (4.20). That is, they are now transformed to

$$\boxed{\begin{aligned}\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} \\ \nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0}\end{aligned}} \quad (4.25)$$

Another possible gauge is the so-called **Coulomb gauge** where

$$\nabla \cdot \mathbf{A} = 0. \quad (4.26)$$

Then the differential equations for the potentials become

$$\begin{aligned}\nabla^2 \Phi &= -\frac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t}.\end{aligned} \quad (4.27)$$

### 4.3 Retarded Solution for the Scalar and Vector Potentials

#### 4.3.1 First solution

The differential equations (4.25) simplify to already know equations (i.e., equations (1.89) and (3.28)), with their respective solutions (equations (1.57) and (3.30)), for static fields. To find corresponding solutions in the case of varying fields, we will divide space in infinitesimally small volumes and determine the fields induced by a charge located in one of the volume elements, and, in the end, sum over all elements.

We concentrate on solving the second of equations (4.25). The charge  $\Delta q$  contained in one volume element will in general depend on time, and be responsible for the presence of a potential  $\Delta \Phi$ . If we place the origin of the coordinate system within the element, the charge density will be

$$\rho = \Delta q(t) \delta(\mathbf{R}), \quad (4.28)$$

where  $\mathbf{R}$  correspond to the position relative to the origin. We, therefore, seek to solve the following equation

$$\nabla^2 (\Delta \Phi) - \frac{1}{c^2} \frac{\partial^2 (\Delta \Phi)}{\partial t^2} = -\frac{1}{\epsilon_0} \Delta q(t) \delta(\mathbf{R}). \quad (4.29)$$

Everywhere, but at the origin,  $\delta(\mathbf{R}) = 0$  and

$$\nabla^2 (\Delta \Phi) - \frac{1}{c^2} \frac{\partial^2 (\Delta \Phi)}{\partial t^2} = 0, \quad \text{for } \mathbf{R} \neq 0. \quad (4.30)$$

It is apparent that in this case, the potential  $\Phi$  is spherically symmetric and, therefore, depends only on  $R = |\mathbf{R}|$ . Equation (4.30) can then be solved using spherical coordinate, and in substituting  $\Delta\Phi = \chi(R,t)/R$  we have

$$\frac{\partial^2 \chi}{\partial R^2} - \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} = 0. \quad (4.31)$$

The solution to equation (4.31) (the wave equation) is simply

$$\chi = \chi\left(t \pm \frac{R}{c}\right). \quad (4.32)$$

Since an outgoing wave emanating from the source (i.e., charge distribution) is the most likely physical situation, we will limit ourselves to the solution

$$\Delta\Phi = \frac{\chi\left(t - \frac{R}{c}\right)}{R}. \quad (4.33)$$

We still have to specify the form of the function  $\chi$ , while ensuring that the potential  $\Delta\Phi$  behaves satisfactorily at the origin. We know that in the limit where  $R \rightarrow 0$  the right-hand side of equation (4.29) goes to infinity. We also see from equation (4.33) that the potential also tends to infinity when  $R \rightarrow 0$ . This implies that its derivatives relative to the coordinates vary much faster than its time derivatives. We therefore neglect the term involving  $\partial^2(\Delta\Phi)/\partial t^2$  in equation (4.29), and get

$$\nabla^2(\Delta\Phi) = -\frac{1}{\epsilon_0} \Delta q(t) \delta(\mathbf{R}), \quad \text{when } \mathbf{R} \rightarrow 0. \quad (4.34)$$

Equation (4.34) is similar to the Poisson equation for a point charge, and we can write

$$\lim_{R \rightarrow 0} \Delta\Phi = \frac{\Delta q(t)}{4\pi\epsilon_0 R}, \quad (4.35)$$

and, in general, when combining equations (4.33) and (4.35)

$$\Delta\Phi = \frac{\Delta q\left(t - \frac{R}{c}\right)}{4\pi\epsilon_0 R}. \quad (4.36)$$

If we now set  $\Delta q = \rho dV$  and integrate over all space, we find the solution to be

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad (4.37)$$

where we set  $R = |\mathbf{x} - \mathbf{x}'|$ . Following a similar treatment, we would derive a corresponding solution for the vector potential. We finally write the solutions

$$\boxed{\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{[\rho(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ \mathbf{A}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \int \frac{[\mathbf{J}(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|} d^3x' \end{aligned}} \quad (4.38)$$

where the square bracket  $[\dots]_{\text{ret}}$  means that the time  $t'$  is to be evaluated at the retarded time  $t' = t - |\mathbf{x} - \mathbf{x}'|/c$ .

#### 4.3.2 Second solution

The last of equations (4.25) can be solved more formally and in a mathematically more elegant manner by using Laplace and Fourier transformations. To do so we set the problem within the context of the Green function formalism. That is, we first try to solve the following second order differential equation

$$\nabla^2 G(\mathbf{x} - \mathbf{x}', t - t') - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\mathbf{x} - \mathbf{x}', t - t') = -\delta(\mathbf{x} - \mathbf{x}', t - t'), \quad (4.39)$$

and then evaluate the potential  $\Phi(\mathbf{x}, t)$  with the following convolution integral

$$\Phi(\mathbf{x}, t) = \frac{1}{\epsilon_0} \int \rho(\mathbf{x}', t') G(\mathbf{x} - \mathbf{x}', t - t') d^3x' dt'. \quad (4.40)$$

Using the three-dimensional equivalent to equations (2.8) for the Fourier transform linking the  $\mathbf{z} \equiv \mathbf{x} - \mathbf{x}'$  and  $\mathbf{k}$  domains, and that for the (two-sided) Laplace transform linking the  $\tau \equiv t - t'$  and  $s$  domains, we find that the transformation for equation (4.39) is

$$G(\mathbf{k}, s) \left( -k^2 - \frac{s^2}{c^2} \right) = -\frac{1}{(2\pi)^{3/2}}, \quad (4.41)$$

or



$$\begin{aligned}
G(\mathbf{k}, s) &= \frac{1}{(2\pi)^{3/2}} \frac{c^2}{s^2 + k^2 c^2} \\
&= \frac{1}{(2\pi)^{3/2}} \frac{c^2}{(s + ikc)(s - ikc)}.
\end{aligned} \tag{4.42}$$

We now need to calculate the inverse Laplace and Fourier transforms of equation (4.42). Starting with the former we must be careful to use the relations that hold for the so-called two-sided Laplace transform (where the limits of integration span the domain  $-\infty < \tau < \infty$ ), since the charge density can exist for times that can be either negative or positive. Taking this into account, we calculate (using a generalization of equation (I.16) for the residue theorem when  $|\tau| > 0$ , for example) the inverse Laplace transform to yield

$$G(\mathbf{k}, \tau) = \frac{1}{(2\pi)^{3/2}} \frac{c}{k} [\sin(kc\tau)H(\tau) - \sin(kc\tau)H(-\tau)], \tag{4.43}$$

where  $H(\tau)$  is the Heaviside distribution introduced in Chapter 1 (see equation (1.14)). Then with a generalization of equations (2.8) the inverse Fourier transform is

$$G(\mathbf{z}, \tau) = \frac{c}{(2\pi)^3} [H(\tau) - H(-\tau)] \int \frac{1}{k} \sin(kc\tau) e^{ik \cdot \mathbf{z}} d^3k. \tag{4.44}$$

We can express the integral in equation (4.44) differently by using the spherical coordinates in  $\mathbf{k}$ -space (such that  $d^3k = k^2 \sin(\theta) dk d\theta d\phi$ )

$$\begin{aligned}
\int \frac{1}{k} \sin(kc\tau) e^{ik \cdot \mathbf{z}} d^3k &= 2\pi \int_0^\infty dk k \sin(kc\tau) \int_{-1}^1 d[\cos(\theta)] e^{-ikz \cos(\theta)} \\
&= \frac{4\pi}{z} \int_0^\infty dk \sin(kc\tau) \sin(kz) \\
&= \frac{\pi}{z} \int_0^\infty dk [e^{ik(z-c\tau)} + e^{-ik(z-c\tau)} - e^{ik(z+c\tau)} - e^{-ik(z+c\tau)}].
\end{aligned} \tag{4.45}$$

We can further transform this one-sided integral into a two-sided integral by making the change of variable  $k \rightarrow -k$  for the two terms where the exponent shows a leading minus sign, and pairing the corresponding terms. We thus get

$$\begin{aligned}
G(\mathbf{z}, \tau) &= \frac{c}{8\pi^2 z} [H(\tau) - H(-\tau)] \int_{-\infty}^\infty dk [e^{ik(z-c\tau)} - e^{ik(z+c\tau)}] \\
&= \frac{c}{4\pi z} [H(\tau) - H(-\tau)] [\delta(z - c\tau) - \delta(z + c\tau)].
\end{aligned} \tag{4.46}$$

However, because  $z = |\mathbf{z}| > 0$  and the known definitions for the Dirac and Heaviside distributions, we have the following simplifications

$$\begin{aligned} H(\tau)\delta(z - c\tau) &= \delta(z - c\tau) \\ H(\tau)\delta(z + c\tau) &= 0, \end{aligned} \quad (4.47)$$

and

$$\begin{aligned} H(-\tau)\delta(z - c\tau) &= 0 \\ H(-\tau)\delta(z + c\tau) &= \delta(z + c\tau). \end{aligned} \quad (4.48)$$

We, therefore, find that the Green function is

$$G(\mathbf{z}, \tau) = \frac{c}{4\pi z} [\delta(z - c\tau) + \delta(z + c\tau)]. \quad (4.49)$$

The two terms in equation (4.49) are often treated independently, each represented by its own Green function

$$G^\pm(\mathbf{z}, \tau) = \frac{c}{4\pi z} \delta(z \mp c\tau), \quad (4.50)$$

where  $G^\pm(\mathbf{z}, \tau)$  are Green functions that physically corresponds to either a “retarded” wave (for the ‘+’ sign) or an “advanced” wave (for the ‘-’ sign). The retarded solution is the most common physical situation. Finally, inserting equation (4.50) into equation (4.40) we get

$$\begin{aligned} \Phi^\pm(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t') \delta[|\mathbf{x} - \mathbf{x}'| \mp c(t - t')]}{|\mathbf{x} - \mathbf{x}'|} d^3x' cd t' \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\mathbf{x}', t \mp \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \end{aligned} \quad (4.51)$$

which is the same result as that of equation (4.37) obtained earlier (restricting ourselves to the retarded solution).

#### 4.4 Retarded Solution for the Fields

We can also derive wave equations for the electric and the magnetic induction fields. For example, by taking the curl of the equation for Faraday’s law (i.e., the last of equations (4.5)) we get

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}),\end{aligned}\tag{4.52}$$

and upon using the first and third of equations (4.5), with  $\mathbf{B} = \mu_0 \mathbf{H}$  and  $\mathbf{D} = \epsilon_0 \mathbf{E}$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0} \left( \nabla \rho + \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t} \right).\tag{4.53}$$

Similarly (i.e., using Maxwell's equations), we find that

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \mathbf{J}.\tag{4.54}$$

The general solution to equations (4.53) and (4.54) is obtained through the same technique used for the potentials, and

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} d^3 x' \\ \mathbf{B}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \int \frac{1}{R} [\nabla' \times \mathbf{J}]_{\text{ret}} d^3 x',\end{aligned}\tag{4.55}$$

where  $R = |\mathbf{x} - \mathbf{x}'|$ , and  $t' = t - R/c$ . These relations can be further modified by explicitly evaluating the spatial derivatives. Doing so, we find

$$\begin{aligned}\nabla' [\rho]_{\text{ret}} &= [\nabla' \rho]_{\text{ret}} + \left[ \frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \nabla' \left( t - \frac{R}{c} \right) \\ &= [\nabla' \rho]_{\text{ret}} + \frac{\mathbf{e}_R}{c} \left[ \frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \\ \nabla' \times [\mathbf{J}]_{\text{ret}} &= [\nabla' \times \mathbf{J}]_{\text{ret}} + \nabla' \left( t - \frac{R}{c} \right) \times \left[ \frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} \\ &= [\nabla' \times \mathbf{J}]_{\text{ret}} - \frac{1}{c} \left[ \frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} \times \mathbf{e}_R,\end{aligned}\tag{4.56}$$

since the retarded quantities (i.e.,  $\rho$  or  $\mathbf{J}$ ) depends on  $\mathbf{x}'$  explicitly, and also implicitly through  $t'$  (via  $R$ ). It should be noted, however, that because in the integrands of equations (4.55)  $\mathbf{x}'$  is not a function of  $t'$  (in fact, the different quantities involved in the integrals are independent of  $t'$ ) and  $t' = t - R/c$ , then

$$\left[ \frac{\partial f(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} = \frac{\partial}{\partial t} [f(\mathbf{x}', t')]_{\text{ret}}. \quad (4.57)$$

Inserting equations (4.56) into equations (4.55) we have

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{1}{R} \left\{ -\nabla' [\rho]_{\text{ret}} + \frac{\mathbf{e}_R}{c} \left[ \frac{\partial \rho}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2} \left[ \frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} \right\} d^3 x' \\ &= \frac{1}{4\pi\epsilon_0} \int \left\{ -\nabla' \left( \frac{[\rho]_{\text{ret}}}{R} \right) + \frac{\mathbf{e}_R}{R^2} [\rho]_{\text{ret}} + \frac{\mathbf{e}_R}{cR} \left[ \frac{\partial \rho}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[ \frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} \right\} d^3 x', \end{aligned} \quad (4.58)$$

where the first term on the right-hand side will vanish after the corresponding volume integral is transformed into a surface integral with equation (1.32). The same can be done for the magnetic induction field (using equation (1.31) instead of equation (1.32), this time), and we find the so-called **Jefimenko equations** for the fields

$$\boxed{\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int \left\{ \frac{\mathbf{e}_R}{R^2} [\rho(\mathbf{x}', t')]_{\text{ret}} + \frac{\mathbf{e}_R}{cR} \left[ \frac{\partial \rho(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[ \frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \right\} d^3 x' \\ \mathbf{B}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \int \left\{ [\mathbf{J}(\mathbf{x}', t')]_{\text{ret}} \times \frac{\mathbf{e}_R}{R^2} + \left[ \frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\mathbf{e}_R}{cR} \right\} d^3 x' \end{aligned}} \quad (4.59)$$

We notice that these relations for the fields reduce to, respectively, equations (1.47) and (3.13) when charge and current distributions are static. As an application of the Jefimenko formulae, we consider the case of a point charge with

$$\begin{aligned} \rho(\mathbf{x}', t') &= q\delta[\mathbf{x}' - \mathbf{r}(t')] \\ \mathbf{J}(\mathbf{x}', t') &= q\mathbf{v}(t')\delta[\mathbf{x}' - \mathbf{r}(t')], \end{aligned} \quad (4.60)$$

where  $\mathbf{r}(t')$  and  $\mathbf{v}(t')$  are the position and velocity of the charge at time  $t'$ , respectively. Substituting equations (4.60) in equations (4.59), and using equation (4.57), we find that (see the second list of problems)

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \frac{q}{4\pi\epsilon_0} \left\{ \left[ \frac{\mathbf{e}_R}{\kappa R^2} \right]_{\text{ret}} + \frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\mathbf{e}_R}{\kappa R} \right]_{\text{ret}} - \frac{1}{c^2} \frac{\partial}{\partial t} \left[ \frac{\mathbf{v}}{\kappa R} \right]_{\text{ret}} \right\} \\ \mathbf{B}(\mathbf{x}, t) &= q \frac{\mu_0}{4\pi} \left\{ \left[ \frac{\mathbf{v} \times \mathbf{e}_R}{\kappa R^2} \right]_{\text{ret}} + \frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\mathbf{v} \times \mathbf{e}_R}{\kappa R} \right]_{\text{ret}} \right\}, \end{aligned} \quad (4.61)$$

with  $\kappa = 1 - \mathbf{e}_R \cdot \mathbf{v}/c$ . Following Heaviside (for the magnetic induction) and Feynman (for the electric field), these equations can be transformed to

$$\boxed{
\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{q}{4\pi\epsilon_0} \left\{ \left[ \frac{\mathbf{e}_R}{R^2} \right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c} \frac{\partial}{\partial t} \left[ \frac{\mathbf{e}_R}{R^2} \right]_{\text{ret}} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} [\mathbf{e}_R]_{\text{ret}} \right\} \\
\mathbf{B}(\mathbf{x},t) &= q \frac{\mu_0}{4\pi} \left\{ \left[ \frac{\mathbf{v} \times \mathbf{e}_R}{\kappa^2 R^2} \right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}} \frac{\partial}{\partial t} \left[ \frac{\mathbf{v} \times \mathbf{e}_R}{\kappa} \right]_{\text{ret}} \right\}
\end{aligned}
} \quad (4.62)$$

Although not extremely difficult, the derivation of these equations is somewhat complicated by a subtle point pertaining to the relationship between the two types of time derivatives (i.e.,  $\partial[f]_{\text{ret}}/\partial t$  and  $[\partial f/\partial t']_{\text{ret}}$ ). More precisely, since after the integration of equations (4.59)  $\mathbf{x}'$  is replaced by  $\mathbf{r}(t')$  and thus acquires a dependency on  $t'$  (which is passed to  $\mathbf{e}_R, R$ , and  $\kappa$ ), then equation (4.57) does not apply anymore. Instead, it is possible to show that

$$\frac{\partial[f]_{\text{ret}}}{\partial t} = \left[ \frac{1}{\kappa} \frac{\partial f}{\partial t'} \right]_{\text{ret}}. \quad (4.63)$$

Diligent application of this formula will allow the derivation of equations (4.62) from equations (4.61).

One last comment should be made about the Heaviside-Feynman fields for a point charge. From the dependency on  $R$  of the different terms, it is easy to see that the dominant ones at large distance from the charge are those proportional to the acceleration (and/or changes in the current) of the charges. This is how electromagnetic radiation is generated by localized charge and current densities.

## 4.5 Poynting's Theorem, and the Conservation Laws

### 4.5.1 The Conservation of Energy

Since the force acting on a point charge  $q$  subjected to electromagnetic fields is given by the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (4.64)$$

then the mechanical work done by the fields on the charge is

$$\frac{dE_{\text{mech}}}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}. \quad (4.65)$$

(The magnetic field does no work.) We can easily generalize equation (4.65) to an arbitrary distribution charge density  $\rho$  using the fact that  $\mathbf{J} = \rho\mathbf{v}$ , then

$$\frac{dE_{\text{mech}}}{dt} = \int_V \mathbf{J} \cdot \mathbf{E} d^3x. \quad (4.66)$$

This equation can be expanded using the Ampère-Maxwell law (i.e., the third of equations (4.5)) to eliminate the current density

$$\int_V \mathbf{J} \cdot \mathbf{E} \, d^3x = \int_V \left[ (\nabla \times \mathbf{H}) \cdot \mathbf{E} - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} \right] d^3x. \quad (4.67)$$

The first term in the right-hand side can be advantageously modified with

$$(\nabla \times \mathbf{H}) \cdot \mathbf{E} = (\nabla \times \mathbf{E}) \cdot \mathbf{H} - \nabla \cdot (\mathbf{E} \times \mathbf{H}), \quad (4.68)$$

and Faraday's law to substitute for  $\nabla \times \mathbf{E}$

$$(\nabla \times \mathbf{H}) \cdot \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H} - \nabla \cdot (\mathbf{E} \times \mathbf{H}). \quad (4.69)$$

Inserting equation (4.69) into equation (4.67) we get

$$\int_V \mathbf{J} \cdot \mathbf{E} \, d^3x = -\int_V \left[ \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] d^3x. \quad (4.70)$$

If we now assume that the medium is linear and isotropic (i.e.,  $\mathbf{D} = \epsilon_0 \mathbf{E}$  and  $\mathbf{B} = \mu_0 \mathbf{H}$ ), with negligible dispersion or losses, and that the total energy contained in the electromagnetic fields is the sum of the corresponding quantities derived for electrostatics and magnetostatics (i.e., equations (2.151) and (3.118)), then the total electromagnetic energy density  $u$  is

$$u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}), \quad (4.71)$$

and equation (4.70) becomes

$$\int_V \mathbf{J} \cdot \mathbf{E} \, d^3x = -\int_V \left[ \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) \right] d^3x. \quad (4.72)$$

Since the volume of integration  $V$  is completely arbitrary, we can write this last equation as a continuity equation that expresses the **law of conservation of energy** or **Poynting's theorem**

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}} \quad (4.73)$$

where we introduced the **Poynting vector**  $\mathbf{S}$  (with units of energy/(area  $\times$  time)) that represents the energy flow

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H}} \quad (4.74)$$

Equation (4.73) states that the time rate of change of electromagnetic energy (density) within a certain volume ( $\partial u / \partial t$ ) plus the energy flowing out of the volume per unit time ( $\nabla \cdot \mathbf{S}$ ), equals the negative of total mechanical work done by the electromagnetic fields on the charges within the volume.

We can start with equation (4.66) to rework the integral formulation of the law of conservation energy (i.e., equation (4.72)). That is,

$$\frac{dE}{dt} = \frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = - \int_S \mathbf{S} \cdot \mathbf{n} \, da, \quad (4.75)$$

where the divergence theorem was used to transform the volume integral containing the Poynting vector to a surface integral. We defined the total energy  $E$  contained within the volume as being the sum of the mechanical energy and the energy contained in the fields, with

$$E_{\text{field}} = \int_V u \, d^3x = \frac{\epsilon_0}{2} \int_V (E^2 + c^2 B^2) \, d^3x. \quad (4.76)$$

#### 4.5.2 The Conservation of Linear Momentum

The Lorentz force (equation (4.64)) is easily generalized to the case of arbitrary charge and current densities with

$$\mathbf{F} = \frac{d\mathbf{P}_{\text{mech}}}{dt} = \int_V (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d^3x, \quad (4.77)$$

where we used Newton's second law to define the total mechanical momentum carried by the charges. We now use the first and third of equations (4.5) to eliminate the charge and current densities from the integrand (still assuming that the medium is linear and isotropic)

$$\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \epsilon_0 \left[ \mathbf{E} (\nabla \cdot \mathbf{E}) + \mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} - c^2 \mathbf{B} \times (\nabla \times \mathbf{B}) \right], \quad (4.78)$$

but since

$$\begin{aligned}
\mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} &= -\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \\
&= -\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}),
\end{aligned} \tag{4.79}$$

and  $c^2 \mathbf{B}(\nabla \cdot \mathbf{B}) = 0$ , we can transform equation (4.78) to

$$\begin{aligned}
\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} &= \varepsilon_0 [\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) \\
&\quad + c^2 \mathbf{B}(\nabla \cdot \mathbf{B}) - c^2 \mathbf{B} \times (\nabla \times \mathbf{B})] - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}).
\end{aligned} \tag{4.80}$$

The first two terms (and therefore the third and fourth as well) on the right-hand side can be simplified as follows

$$\begin{aligned}
[\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E})]_i &= E_i \partial_j E_j - \varepsilon_{ijk} E_j \varepsilon_{kmn} \partial_m E_n \\
&= E_i \partial_j E_j - (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) E_j \partial_m E_n \\
&= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i \\
&= \partial_j (E_i E_j) - \frac{1}{2} \partial_i (E_j E_j) \\
&= \partial_j \left[ (E_i E_j) - \frac{1}{2} \delta_{ij} (E_k E_k) \right].
\end{aligned} \tag{4.81}$$

Using this result, and introducing the **Maxwell stress Tensor**  $T_{ij}$

$$\boxed{T_{ij} = \varepsilon_0 \left[ E_i E_j + c^2 B_i B_j - \frac{1}{2} \delta_{ij} (E_k E_k + c^2 B_k B_k) \right]} \tag{4.82}$$

and the **electromagnetic momentum density vector**  $\mathbf{g}$

$$\boxed{\mathbf{g} \equiv \frac{1}{c^2} (\mathbf{E} \times \mathbf{H}) = \frac{1}{c^2} \mathbf{S}} \tag{4.83}$$

we rewrite equation (4.80) as

$$(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})_i = \partial_j T_{ij} - \left( \frac{\partial \mathbf{g}}{\partial t} \right)_i, \tag{4.84}$$

or



$$\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \nabla \cdot \vec{\mathbf{T}} - \frac{\partial \mathbf{g}}{\partial t}, \quad (4.85)$$

where  $\vec{\mathbf{T}}$  also represents the second rank (and symmetric) Maxwell stress tensor. Defining the electromagnetic momentum  $\mathbf{P}_{\text{field}}$  as

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2} \int_V \mathbf{E} \times \mathbf{H} d^3x = \int_V \mathbf{g} d^3x, \quad (4.86)$$

then, combining equations (4.77), and (4.82)-(4.86), we get

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}}) = \int_V \nabla \cdot \vec{\mathbf{T}} d^3x, \quad (4.87)$$

or alternatively, using the divergence theorem to transform the volume integral into a surface integral, we have

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}}) = \int_S \vec{\mathbf{T}} \cdot \mathbf{n} da, \quad (4.88)$$

where  $\mathbf{n}$  is the outward unit vector normal to the surface  $S$ . It follows that if equations (4.87) and (4.88) are a statement of the conservation of linear momentum, it must be then that the term  $T_{ij}n_j$  corresponds to the flow per unit area of momentum across the surface  $S$  delimiting the volume  $V$ . It can also be interpreted as the force per unit area transmitted across the surface and acting on the system of (charged) particles and electromagnetic fields.

### 4.5.3 Conservation of Angular Momentum

We have, by now, established that the electromagnetic fields carry an energy density

$$u = \frac{\epsilon_0}{2} (E^2 + c^2 B^2), \quad (4.89)$$

and a linear momentum density

$$\mathbf{g} = \frac{1}{c^2} \mathbf{S} = \epsilon_0 (\mathbf{E} \times \mathbf{B}). \quad (4.90)$$

Accordingly, one can define an **angular momentum density** for the electromagnetic fields with

$$\boxed{\mathcal{L}_{\text{field}} = \mathbf{x} \times \mathbf{g} = \frac{1}{c^2} \mathbf{x} \times \mathbf{S} = \frac{1}{c^2} \mathbf{x} \times (\mathbf{E} \times \mathbf{H})} \quad (4.91)$$

Furthermore, if we introduce a second order rank tensor  $\vec{\vec{M}}$  for the flux of angular momentum, then it can be shown that if (see the second problem list)

$$\vec{\vec{M}} = \vec{T} \times \mathbf{x}, \quad (4.92)$$

then

$$\frac{\partial}{\partial t} (\mathcal{L}_{\text{mech}} + \mathcal{L}_{\text{field}}) + \nabla \cdot \vec{\vec{M}} = 0, \quad (4.93)$$

and

$$\frac{d}{dt} \int_V (\mathcal{L}_{\text{mech}} + \mathcal{L}_{\text{field}}) d^3x + \int_S \vec{\vec{M}} \cdot \mathbf{n} da = 0. \quad (4.94)$$

The quantity  $\mathcal{L}_{\text{mech}}$  in equations (4.93) and (4.94) is the mechanical angular momentum density of the charged particles interacting with the electromagnetic fields. It is interesting to note that even static fields can hold linear and angular momenta, as long as the Poynting vector is nonzero.

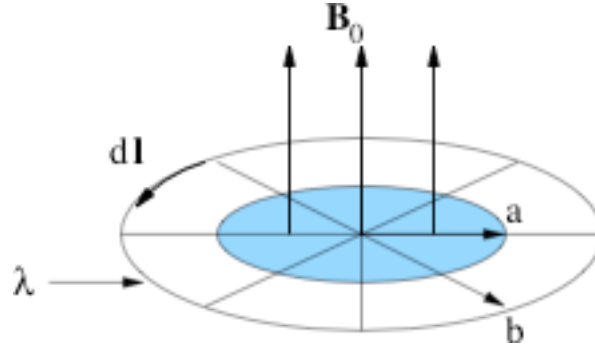
### Example

A line of charge  $\lambda$  is uniformly glued onto the rim of a wheel of radius  $b$ , which is then suspended horizontally, as shown in Figure 4-1. Although it is originally at rest, the wheel is free to rotate (the spokes are made of some non-conducting material – wood, maybe). In the central region, out to a radius  $a (< b)$ , there is initially a uniform magnetic induction  $\mathbf{B}_0$ , pointing up. Now someone turns the field off.

- a) Use Lenz' law, and whatever conservation law(s) that might be needed, to qualitatively describe what happens to the wheel after the magnetic induction field is turned off. Does the wheel stay at rest or is it set in some sort of mechanical motion?
- b) Now, quantify the answer you gave in a), again using the needed laws (either from conservation arguments, and/or electromagnetic considerations). Does the final state of the wheel depend on the details of how  $\mathbf{B}_0$  is turned off?
- c) If you determined that the wheel is set in some sort of motion by the disappearance of the magnetic induction, then specify exactly what was the source of the motion, and the agent responsible for this motion. Be specific.

### Solution.

a) The changing magnetic induction field will induce an electric field (Faraday's law of induction), curling around the axis of the wheel. This electric field exerts a force on the charges glued to the rim, and the wheel starts to turn. According to Lenz' law, it will rotate in such a direction that *its* (magnetic induction) field tends to restore the initial upward flux. The rotational motion, then, is counterclockwise, as viewed from above.



**Figure 4-1** - A line charge  $\lambda$  is uniformly glued to the wheel at a radius  $b$ , and a magnetic induction field  $\mathbf{B}_0$  initially exists out to a radius  $a$ , until it is turned off. The wheel is free to rotate.

b) Quantitatively, Faraday's law states that

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{l} &= -\frac{dF}{dt} \\ &= -\frac{d}{dt} \int_s \mathbf{B} \cdot \mathbf{n} da \\ &= -\pi a^2 \frac{dB}{dt}.\end{aligned}\tag{4.95}$$

The torque on a segment of length  $d\mathbf{l}$  along the rim of the wheel is given by

$$d\mathbf{N} = [\mathbf{r} \times (E \lambda d\mathbf{l})],\tag{4.96}$$

where  $E$  is the induced electric field. Integrating over the rim, we get the total torque on the wheel

$$\begin{aligned}\mathbf{N} &\equiv \frac{d\mathbf{L}}{dt} = \oint \mathbf{x} \times (E \lambda d\mathbf{l}) \\ &= \mathbf{e}_z b \lambda \oint E dl \\ &= -\mathbf{e}_z \lambda b \pi a^2 \frac{dB}{dt}.\end{aligned}\tag{4.97}$$

The total angular momentum of the rim, after the initial magnetic induction has been turned off, is

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{N} dt \\
&= -\mathbf{e}_z \lambda b \pi a^2 \int_{B_0}^0 dB \\
&= \mathbf{e}_z \lambda b \pi a^2 B_0,
\end{aligned} \tag{4.98}$$

and the wheel is turning counterclockwise when seen from above, as stated in **a**). Since we must also have  $\mathbf{L} = I_z \boldsymbol{\omega} \mathbf{e}_z$ , with  $I_z$  the principal moment of inertia of the wheel about its symmetry axis, then

$$\boldsymbol{\omega} = \frac{\lambda \pi a^2 b}{I_z} B_0. \tag{4.99}$$

That is, the speed of the wheel is independent of the way in which  $\mathbf{B}_0$  is turned off.

**c)** There must be conservation of angular momentum for this system. Since the wheel has acquired a net quantity of angular momentum, the electromagnetic fields must have lost an equal amount. So, *the source of the motion was the initial angular momentum present in the electromagnetic fields, and the agent for the motion of the source is the electric field induced by the cancellation of  $\mathbf{B}_0$ .*